Math 259A Lecture 10 Notes

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1 Sups and Infs of Self-Adjoint Operators

1.1 Sups and infs of self-adjoint operators

For $x \in B(H)$, we defined the left support $\ell(x) = [xH]$ and the right support $r(x)$ as the projection onto $(\ker x)^\perp$. We had that $\ell(x) = r(x^*)$ and $\ell(x^*) = r(x)$. So if $x = x^*$, then we can define $\ell(x) = r(x) =: s(x)$, the support of x. We also had the following:

Proposition 1.1. *The left and right support satisfy the following:*

1. *$\ell(x)$ is the smallest projection $e \in B(H)$ such that $ex = x$.*
2. *$r(x)$ is the smallest projection $f \in B(H)$ such that $xf = x$.*

Definition 1.1. If $\{e_i\}$ is a family of projections in $\mathcal{B}(H)$, we denote by $\bigvee_i e_i$ the orthogonal projection onto $\overline{\operatorname{span}\{\operatorname{im} e_i\}}$. Denote by $\bigwedge_i e_i$ the orthogonal projection onto $\bigcap_i \operatorname{im} e_i$

Proposition 1.2. *$\bigvee_i e_i$ is the smallest projection e in $B(H)$ such taht $e \geq e_i$ for all i. $\bigwedge_i e_i$ is the largest projection e in $\mathcal{B}(M)$ such that $e \leq e_i$ for all i.*

Proposition 1.3. *If $\{x_i\} \subseteq \mathcal{B}(H)_h$ is uniformly bounded ($\sup_i \|x_i\| < \infty$), then there is a unique $x = x^* \in \mathcal{B}(H)$ such that $x \geq x_i$ for all i and such that if $y = y^* \geq x_i$ for all i, then $y \leq x$. Moreover, if $\{x_i\}$ is an increasing net ($i \leq j \implies x_i \leq x_j$), then $x_i \xrightarrow{\text{so}} x$.*

Remark 1.1. This says that there is a least upper bound $\sup_i x_i$ of $\{x_i\}$ in $\mathcal{B}(H)_h$. Similarly, there exists some $\inf_i x_i$.

Proof. We can assume $0 \leq x_i \leq 1$; if $K = \sup_i \|x_i\|$, then $1 \geq \frac{1}{2K}(x_i + K\mathbf{1}) \geq 0$. For $\xi \in H$, denote $F(\xi, \xi) = \sup_i \langle x_i \xi, \xi \rangle$. Then define $F(\xi, \eta)$ by polarization:

$$F(\xi, \eta) = \frac{1}{4} \sum_{i=0}^{3} i^k F(\xi + i^k \eta, \xi + i^k \eta).$$

Then $|F(\xi,\eta)| \leq \|\xi\|\|\eta\|$ means F is bounded. By the Riesz representation theorem, there is a unique $x \in \mathcal{B}(H)$ such that $\|x\| \leq 1$ and $x \geq 0$ such that $F(\xi, eta) = \langle x\xi, \eta\rangle$ for all $\xi, \eta \in H$. So $\langle x, \xi, \xi\rangle = \sup_i \langle x_i\xi, \xi\rangle$.

To get $x_i \xrightarrow{\text{so}} x$, we want $\|(x - x_i)\xi\| \to 0$ for all $\xi \in H$. We have by functional calculus that

$$\|(x - x_i)\xi\|^2 = \|(x - x_i)^{1/2}(x - x_i)^{1/2}\xi\|^2 \leq \underbrace{\|(x - x_i)^{1/2}\|^2}_{\leq \|x\|} \underbrace{\langle (x - x_i)\xi, \xi\rangle}_{\to 0}.$$

So $x_i \xrightarrow{\text{so}} x$. $\square$

Proposition 1.4. *If e is an orthogonal projection,* $\operatorname{Spec}(e) \subseteq \{0, 1\}$.

Proof. Since $e = e^*$, $\operatorname{Spec}(e) \subseteq \mathbb{R}$. Since $e^2 = e$, we must have $\operatorname{Spec}(e) \subseteq \{0, 1\}$. $\square$

Proposition 1.5. *Let $\{e_i\}$ be a family of projections. Then*

$$\bigvee_i e_i = \sup_i e_i, \qquad \bigwedge_i e_i = \inf_i e_i.$$

Proposition 1.6. *Let $\{e_i\}$ be a family of projections. Then*

$$\bigvee_i e_i = \bigvee_{\substack{J \subseteq I \\ J \text{ finite}}} e_J, \qquad e_J = s\left(\sum_{i \in J} e_i\right),$$

and as J increases, $e_J \nearrow \bigvee_i e_i$. In particular, if $|I| < \infty$, then $\bigvee_i e_i = s(\sum_{i \in I} e_i)$.

Remark 1.2. This says that $(\mathcal{P}(B(H)), \leq)$, the projections on H with $\leq$, is a complete lattice.

1.2 Consequences in von Neumann algebras

Proposition 1.7. *If M is a C^*-algebra with unit, then any $x \in M$ is a linear combinations of 4 unitary elements in M. In other words, $M = \operatorname{span} U(M)$.*

Proof. We have $x = \operatorname{Re} x + i \operatorname{Im} x$. But if $a = a^* \in (M)_1$, then we can view it as a function using functional calculus. Then we can split it up into the sum of $t \mapsto t + i\sqrt{1 - t^2}$ and $t \mapsto t - i\sqrt{1 - t^2}$, which are unitary because their ranges are subsets of the unit circle. $\square$

If $M = M^*$, then $[M\xi] \in M'$ for all $\xi \in H$. So if M is a von Neumann algebra, then $[M'\xi] \in M'' = M$. So to check that $x \in M$, it is necessary and sufficient to check that $u'x(u')^* = x$ for all $u \in U(M')$.

Corollary 1.1. *Let M be a von Neumann algebra. Then $\ell(x), r(x) \in M$.*

We will prove this next time. Here is a consequence.

Corollary 1.2. *Let M be a von Neumann algebra. If $x \in M$ and $x = va$ is the polar decomposition, then $v, a \in M$.*

Proof. For any $u' \in U(M')$, we have $u'x(u')^* = x$. On the other hand, $x = u'va(u')^* = u'v(u')^*u'a(u')^*$. Then $v_0 = u'v(u')^*$ is a partial isometry and $a_0 = u'a(u')^* \geq 0$. Then $a = (x^*x)^{1/2} \in M$. So we just need to show that $v \in M$. We have that $r(v_0) = u'r(v)(u')^* = u'r(x)(u')^*$. By uniqueness of the polar decomposition of x, $v = v_0 \in M$. $\square$

Corollary 1.3. *Let M be a von Neumann algebra. If $\{x_i\} \subseteq M$ is uniformly bounded and increasing, then $\sup_i x_i \in M$.*

This is because $x_i \uparrow \sup_i x$ in the SO-topology.

Corollary 1.4. *Let M be a von Neumann algebra. Then $\mathcal{P}(M)$, the projections in M form a complete lattice.*